Graduate Seminar on Discrete Optimization (S4C1)
Matrix and Operator Scaling

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Matrix and operator scaling

- Matrix scaling Given a matrix $A \in \mathbb{R}^{n\times n}$, find diagonal matrices $L, R \in \mathbb{R}^{n\times n}$ such that LAR is doubly stochastic, i.e., all row and column sums are 1.
- Sinkhorn algorithm Alternate between rescaling all rows to 1 and all columns to 1
- Doubly stochastic operators Given matrices $A_1, A_2, \ldots, A_m \in \mathbb{C}^{n\times n}$ define operators $T, T^* : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$

$$T(X) = \sum_{i=1}^{m} A_i X A_i^\dagger , \quad T^*(X) = \sum_{i=1}^{m} A_i^\dagger X A_i .$$

  T is doubly stochastic if $T(I_n) = T^*(I_n) = I_n$.
- Operator scaling Given $A_1, A_2, \ldots, A_m \in \mathbb{C}^{n\times n}$, find invertible matrices $L, R \in \mathbb{C}^{n\times n}$ such that $(LA_1R, LA_2R, \ldots, LA_mR)$ define a doubly stochastic operator.

Matrix and operator scaling

- Deep connections to matchings, matroids, and other topics in discrete optimization.
- Matrix scaling We will study Sinkhorn scaling and other algorithms and applications
- Operator scaling We will study the generalization of Sinkhorn scaling, and breakthrough applications including:
  - Solving the noncommutative Edmonds problem, i.e. computing the rank of $x_1 A_1 + x_2 A_2 + \ldots + x_m A_m$
  - Computing constants for Brascamp–Lieb inequalities

Structure of seminars

Each seminar session is structured as follows:

1. First part of the talk (10-20 minutes)
   Introduce the topic of the talk.
   Explain what the main goal or main result will be.
   Give some motivation and provide some context — why is the result interesting/relevant?
2. Questions
   One or two (multiple-choice) questions from the speaker to the audience.
   Answer questions from the audience.

Structure of seminars

Each seminar session is structured as follows:

1. First part of the talk (10-20 minutes)
2. Questions
3. Second part of the talk (55-65 minutes) Present proofs, but focus on the main ideas rather than detailed calculations.
4. Discussion
   Questions from the audience.

Parts 1 and 3 must not take more than 75 minutes in total. Recall definitions and results from previous talks when you use them.

What we expect

- Prepare a talk on your assigned topic, including questions for the audience.
- Prepare a 1-2 pages summary containing the most important results and definitions.
- Give an approval talk approximately 2-3 weeks before your talk.
- Participate actively in the discussions during the seminar.
- In addition to reading the assigned paper or sections, it might be necessary to look into other parts of the paper or other sources.

List of papers and topics

1. Analysis of Sinkhorn scaling
   Chakrabarty, D., & Khanna, S. (2021). Better and simpler error analysis of the Sinkhorn-Knopp algorithm for matrix scaling. Mathematical Programming, 188(1), 395–407.
2. Polynomial algorithm via the ellipsoid method
   Kalantari, B., & Khachiyan, L. (1996). On the complexity of nonnegative-matrix scaling. Linear Algebra and its applications, 240, 87–103.
3. Formulation as separable convex flow problem
   Rote, G., & Zachariasen, M. (2007). Matrix scaling by network flow. In SODA 2007

List of papers and topics

4. Strongly polynomial algorithm and permanent approximation
   Linial, N., Samorodnitsky, A., & Wigderson, A. (2000). A Deterministic Strongly Polynomial Algorithm for Matrix Scaling and Approximate Permanents. Combinatorica, 20(4), 545–568.
5. Overview of some applications of matrix scaling
   Idel, M. (2016). A review of matrix scaling and Sinkhorn's normal form for matrices and positive maps. arXiv preprint arXiv:1609.06349, Section 8 plus references.

List of papers and topics

6. Operator scaling and capacity bounds
   Garg, A., Gurvits, L., Oliveira, R., & Wigderson, A. (2020). Operator scaling: theory and applications. Foundations of Computational Mathematics, 20(2), 223-290.
   3 talks
7. Application to Brascamp–Lieb inequalities
   Garg, A., Gurvits, L., Oliveira, R., & Wigderson, A. (2018). Algorithmic and optimization aspects of Brascamp–Lieb inequalities, via operator scaling. Geom. Funct. Anal, 28, 100–145.
   2 talks

Topic assignment and registration

- Website includes these slides, papers, and assignment: https://www.laszlovegh.eu/scaling-seminar
- Tentative date for first talk: Friday, 20 November
- If you would like to participate, send an email to László Végh (lvegh@uni-bonn.de) indicating your name and topic preferences by Friday 11 October.
- We will inform you by email about the assignment of topics.
- Every participant will also be assigned a supervisor that can help with questions.
- After the assignment of topics, you have 1 week to sign your binding registration.
- In addition, all participants must register via BASIS.