Graduate Seminar on Algorithms and Optimization (S4C3)

Fair Division

László Végh
Wenzheng Li
Hannaneh Akrami

Fair Division

Divide **items** among **agents** in a **fair** manner.

Applications:

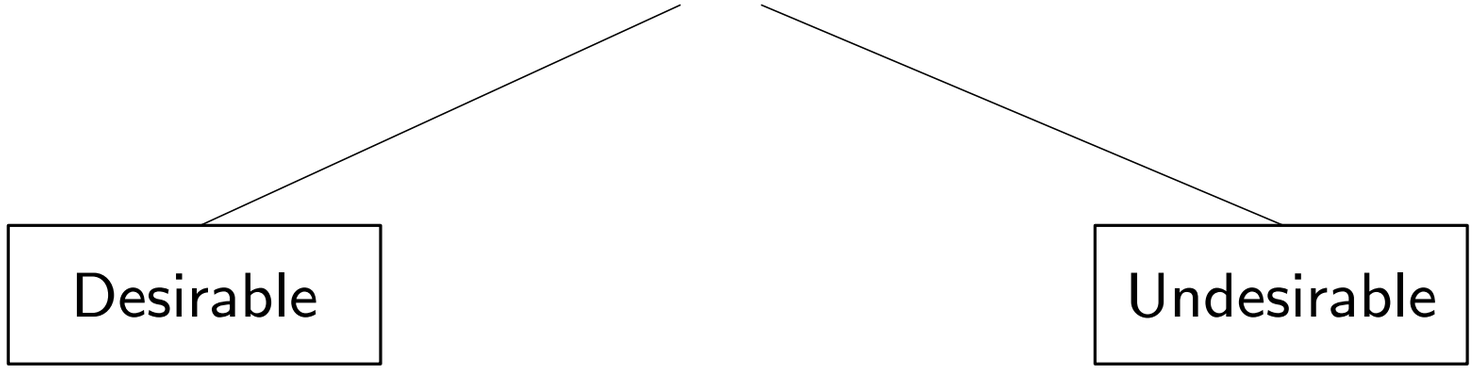
Partnership dissolution

Divorce settlements

Household chores

Air traffic management

Items



Items

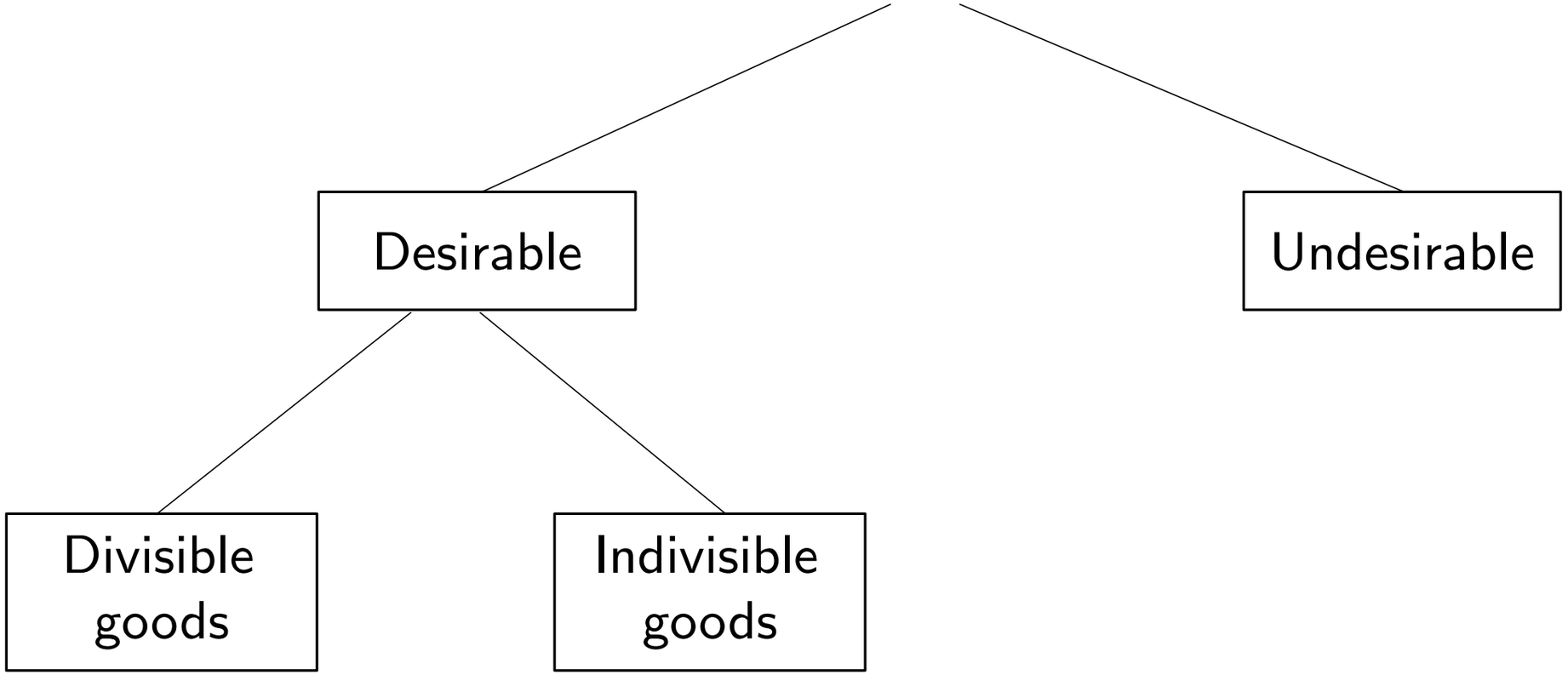
Desirable

Divorce settlements

Undesirable

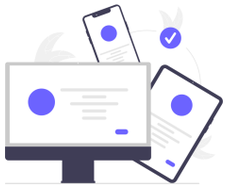
Household chores

Items


Desirable
Undesirable
Divisible goods
Indivisible goods

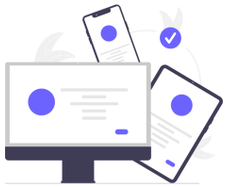
Items

- Desirable
  - Divisible goods
  - Indivisible goods
- Undesirable



Items

- Desirable
  - Divisible goods
  - Indivisible goods
- Undesirable
  - Divisible chores
  - Indivisible chores



Items

- Desirable
  - Divisible goods
  - Indivisible goods
- Undesirable
  - Divisible chores
  - Indivisible chores




Discrete Fair Division

Divide **indivisible items** among **agents** in a **fair** manner.

Input: $\mathcal{I} = (N, M, V)$

- N: set of n agents
- M: set of m indivisible items
- Valuation functions $v_i : 2^M \to \mathbb{R}$

	🍌	🍎	🍓	🍊	🍑
	4	1	2	2	2
	1	0	5	1	1
	1	1	5	1	1

Discrete Fair Division

Divide **indivisible items** among **agents** in a **fair** manner.

Input: $\mathcal{I} = (N, M, V)$

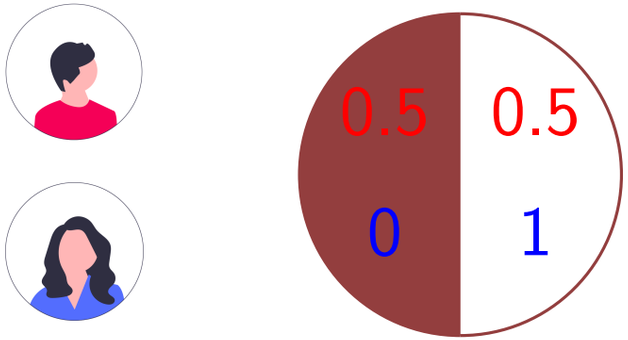
- N: set of n agents
- M: set of m indivisible items
- Valuation functions $v_i : 2^M \to \mathbb{R}$

	🍌	🍎	🍓	🍊	🍑
	4	1	2	2	2
	1	0	5	1	1
	1	1	5	1	1

Goal: Find a **fair** allocation of the items to the agents.

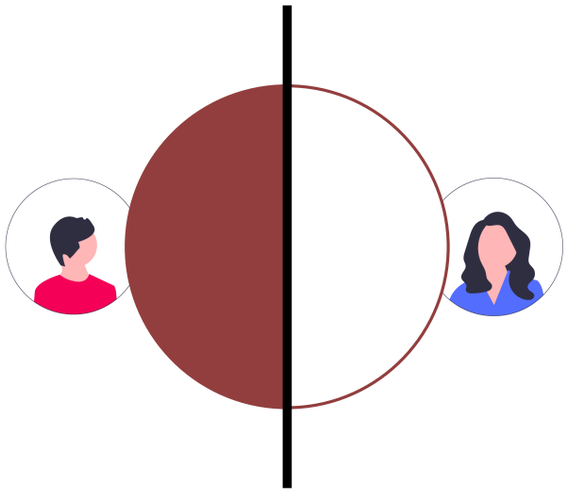
A partition $X = (X_1, X_2, \ldots, X_n, P)$ of M

Fairness

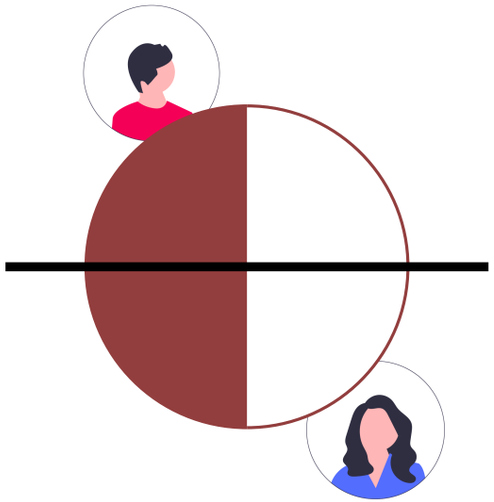


Which allocation is fair?

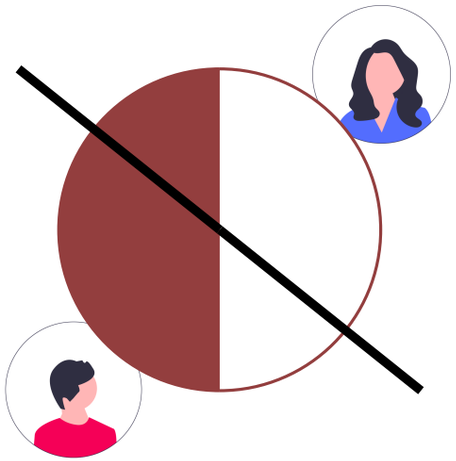
1.



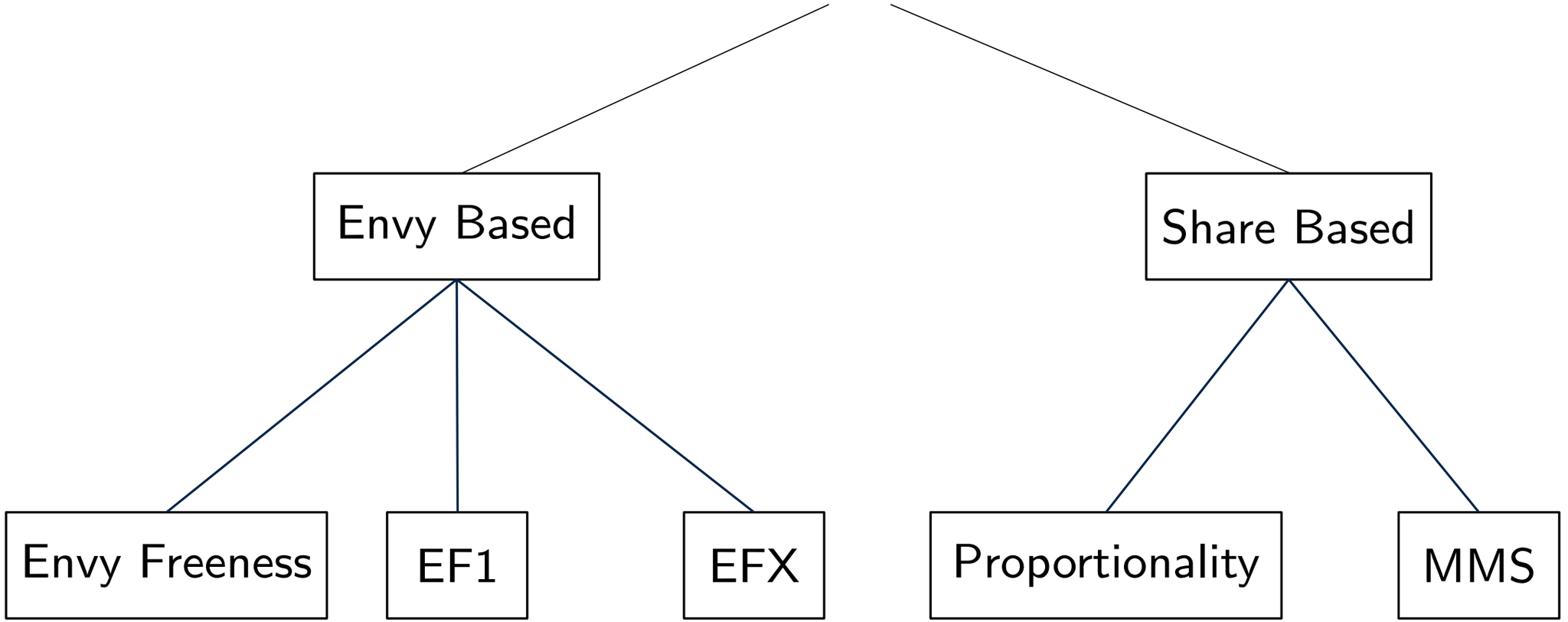
2.



3.

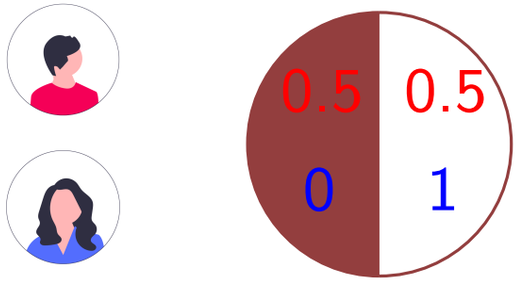


Fairness


Envy Based
Share Based
Envy Freeness
EF1
EFX
Proportionality
MMS

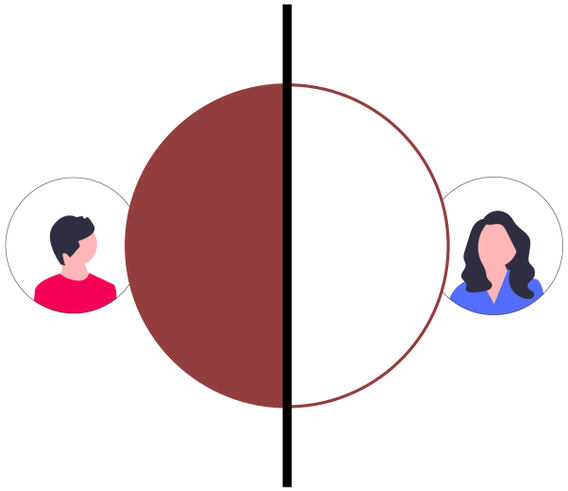
Envy Freeness

Definition: An allocation X is **envy free**, if and only if for all agents i, j: $v_i(X_i) \geq v_i(X_j)$. [Foley 1967]

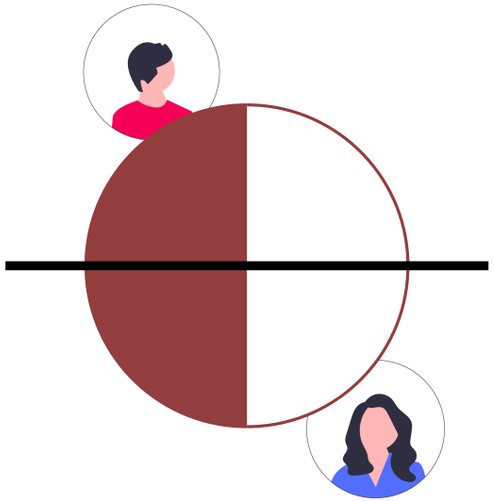


Which allocation is envy free?

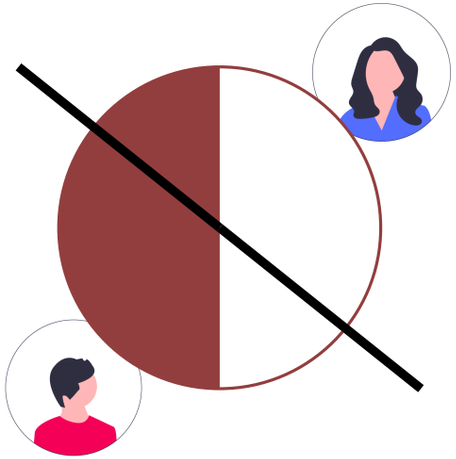
1.



2.



3.



Envy Freeness

Definition: An allocation X is **envy free**, if and only if for all agents i, j: $v_i(X_i) \geq v_i(X_j)$. [Foley 1967]

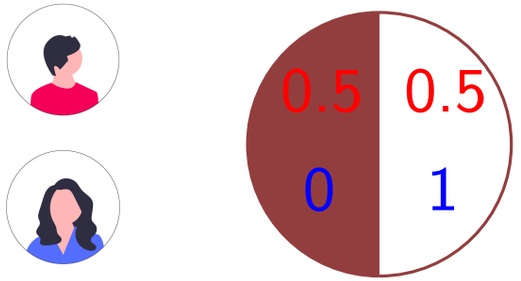
Do envy free allocations always exist?

- For divisible goods, YES!
- For indivisible goods, NO!

Focus has moved to relaxations of envy-freeness: EF1 and EFX.

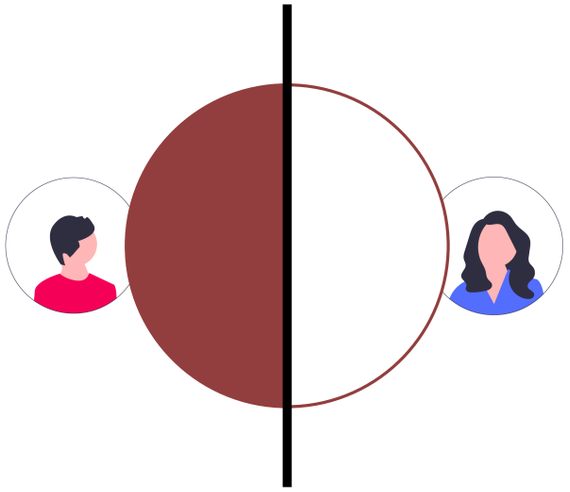
Proportionality

Definition: An allocation X is **proportional**, if and only if for all agents i: $v_i(X_i) \geq v_i(M)/n$.

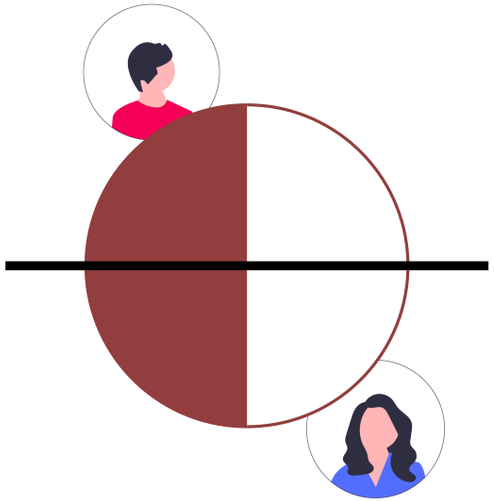


Which allocation is proportional?

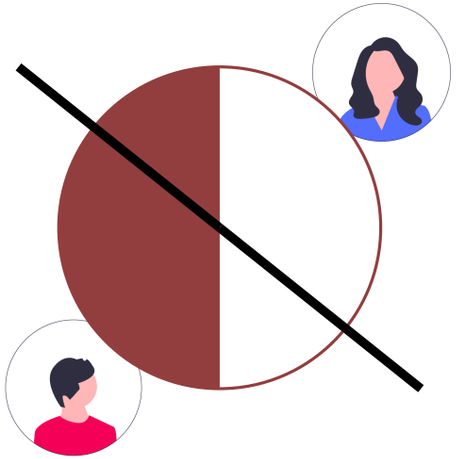
1.



2.



3.



Proportionality

Definition: An allocation X is **proportional**, if and only if for all agents i: $v_i(X_i) \geq v_i(M)/n$.

Do proportional allocations always exist?

- For divisible goods, YES!
- For indivisible goods, NO!

Focus has moved to relaxations of proportionality: Maximin share (MMS)

Efficiency

Divide **items** among **agents** in a **fair** and **efficient** manner.

Definition: Allocation X **pareto dominates** allocation Y, if and only if

- for all agents i, $v_i(X_i) \geq v_i(Y_i)$, and
- there exists an agent j, such that $v_j(X_j) > v_j(Y_j)$.

Definition: Allocation X is **pareto optimal** or **PO** if there exists no allocation Y such that Y pareto dominates X.

Efficiency

Divide **items** among **agents** in a **fair** and **efficient** manner.

Definition: Allocation X **pareto dominates** allocation Y, if and only if

- for all agents i, $v_i(X_i) \geq v_i(Y_i)$, and
- there exists an agent j, such that $v_j(X_j) > v_j(Y_j)$.

Definition: Allocation X is **pareto optimal** or **PO** if there exists no allocation Y such that Y pareto dominates X.

		
	100	1
	1	100

Is this allocation pareto optimal?

Efficiency

Divide **items** among **agents** in a **fair** and **efficient** manner.

Definition: Allocation X **pareto dominates** allocation Y, if and only if

- for all agents i, $v_i(X_i) \geq v_i(Y_i)$, and
- there exists an agent j, such that $v_j(X_j) > v_j(Y_j)$.

Definition: Allocation X is **pareto optimal** or **PO** if there exists no allocation Y such that Y pareto dominates X.

		
	100	1
	1	100

Is this allocation pareto optimal?

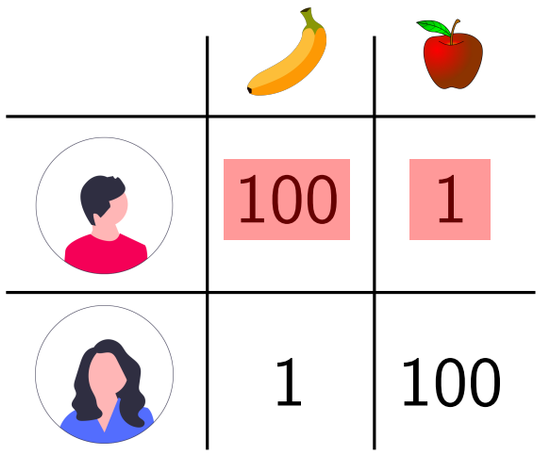
Efficiency

Divide **items** among **agents** in a **fair** and **efficient** manner.

Definition: Allocation X **pareto dominates** allocation Y, if and only if

- for all agents i, $v_i(X_i) \geq v_i(Y_i)$, and
- there exists an agent j, such that $v_j(X_j) > v_j(Y_j)$.

Definition: Allocation X is **pareto optimal** or **PO** if there exists no allocation Y such that Y pareto dominates X.



Is this allocation pareto optimal?

Nash Welfare

Definition: **Nash welfare** of an allocation X is

$$\mathsf{NW}(X) = \left(\prod_{i \in N} v_i(X_i) \right)^{1/n}.$$

Definition: Allocation X is **MNW**, if $\mathsf{NW}(X) \geq \mathsf{NW}(Y)$ for all allocations Y.

Nash Welfare

Definition: Nash welfare of an allocation X is

$$\mathsf{NW}(X) = \left(\prod_{i \in N} v_i(X_i) \right)^{1/n}.$$

Definition: Allocation X is α-**MNW**, if $\mathsf{NW}(X) \geq \alpha \cdot \mathsf{NW}(Y)$ for all allocations Y and $\alpha \in [0, 1]$.

Organization

- Class hours: Fridays 14:15-15:45
- Approval talks: Fridays 16:15-17:45
- Place: Seminar room, Lennéstr. 2
- Prerequisite: Basic familiarity with algorithms and complexity

Structure of Seminars

Each seminar session is structured as follows:

1. First part of the talk (10-20 minutes)
   - Introduce the topic of the talk
   - Explain what the main goal or main result will be.
   - Give some motivation and provide some context – why is the result interesting/relevant?

Structure of Seminars

Each seminar session is structured as follows:

1. First part of the talk (10-20 minutes)

2. Questions

   - One or two (multiple-choice) questions from the speaker to the audience.
   - Questions from the audience.

Structure of Seminars

Each seminar session is structured as follows:

1. First part of the talk (10-20 minutes)
2. Questions
3. Second part of the talk (55-65 minutes)
    - Present proofs, but focus on the main ideas rather than detailed calculations.

Structure of Seminars

Each seminar session is structured as follows:

1. First part of the talk (10-20 minutes)
2. Questions
3. Second part of the talk (55-65 minutes)
4. Discussion
   - Questions from the audience.

Structure of Seminars

Each seminar session is structured as follows:

1. First part of the talk (10-20 minutes)
2. Questions
3. Second part of the talk (55-65 minutes)
4. Discussion

Part 1 and 3 should not take more than 75 minutes in total.
Recall definitions and results from previous talks when you use them.

What we expect

- Prepare a talk on your assigned topic, including questions for the audience.
- Prepare a 1-2 page summary containing the most important results and definitions.
- Give an approval talk 2-3 weeks before your talk.
- Participate actively in the discussions during the seminar.
- In addition to reading the assigned paper or sections, it might be necessary to look into other parts of the paper or other sources.

List of Papers and Topics

1. Rental Harmony: Sperner's Lemma in Fair Division [Su 1999]
   Amer. Math. Monthly, 106(1999), 930-942
   (existence of EF for cake)

2. The Unreasonable Fairness of Maximum Nash Welfare [Caragiannis, Kurokawa, Moulin, Procaccia, Shah, Wang 2016]
   ACM Transactions on Economics and Computation (TEAC)
   (MNW $\implies$ EF1+PO)

3. Convex Program Duality, Fisher Markets, and Nash Social Welfare [Cole, Devanur, Gkatzelis, Jain, Mai, Vazirani, Yazdanbod 2017]
   Proceedings of the 2017 ACM Conference on Economics and Computation (EC'17)
   (2-MNW)

List of Papers and Topics

4. Finding Fair and Efficient Allocations [Barman, Krishnamurthy, Vaish 2018]
   Proceedings of the 2018 ACM Conference on Economics and Computation (EC'18)
   (1.45-MNW + EF1 + PO)

5. On Approximate Envy-Freeness for Indivisible Chores and Mixed Resources [Bhaskar, Sricharan, Vaish 2021]
   Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM'21)
   (EF1 for chores)

List of Papers and Topics

6. A Little Charity Guarantees Almost Envy-Freeness [Chaudhury, Kavitha, Mehlhorn, Sgouritsa 2021]
   SIAM Journal on Computing . 50(4):1336-1358
   ("efficient" partial EFX allocation)

7. EFX: A Simpler Approach and an (Almost) Optimal Guarantee via Rainbow Cycle Number [Akrami, Alon, Chaudhury, Garg, Mehlhorn, Mehta 2024]
   Operations Research
   (EFX for 3 agents)

List of Papers and Topics

8. Simplification and Improvement of MMS Approximation [Akrami, Garg, Sharma, Taki 2023]
   Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence (IJCAI'23)
   ($3/4$-MMS with simple analysis)

9. A Reduction from Chores Allocation to Job Scheduling [Huang, Segal-Halevi 2023]
   Proceedings of the 24th ACM Conference on Economics and Computation (EC'23)
   ($11/13$-MMS for chores)

List of Papers and Topics

10. Approximating Nash Social Welfare by Matching and Local Search [Garg, Husić, Li, Végh, Vondrák 2022]
    Proceedings of the 55th Annual ACM Symposium on Theory of Computing (STOC 2023)
    ($(4+\varepsilon)$-MNW for submodular valuations)

11. A Note on Approximating Weighted Nash Social Welfare with Additive Valuations [Feng, Li 2024]
    51st International Colloquium on Automata, Languages, and Programming (ICALP 2024)
    (1.45-MNW for the weighted additive setting)

List of Papers and Topics

12. Best of Both Worlds: Ex-Ante and Ex-Post Fairness in Resource Allocation
    [Freeman, Shah, Vaish 2020]
    Proceedings of the 24th ACM Conference on Economics and Computation (EC'20)
    (randomized allocations: ex-ante EF + ex-post EF1)

Topics Assignment and Registration

- Website includes these slides, papers, and assignment: https://www.laszlovegh.eu/fairness-seminar/
- If you would like to participate, send an email to Hannaneh Akrami (hakrami@uni-bonn.de) indicating your name and 3-5 topic preferences by Friday 14 February.
- We will inform you by email about the assignment of topics.
- Every participant will also be assigned a supervisor that can help with questions.
- After the assignment of topics, you have 1 week to confirm your participation.
- In addition, all participants must register via BASIS.