Combinatorial optimization - Structures and Algorithms, *GeorgiaTech, Fall 2011*
Problem set 1

László Végh

September 28, 2011

1. Let $G = (V, E)$ be a planar graph, and $x, y \in V$ two nonadjacent nodes. (A) Prove that there is an orientation of E with the in-degree of x and y being 0 and every other in-degree at most 3. (B) Prove that if G is planar and bipartite, then 3 can be replaced by 2 above.

2. Let us be given a 2-node-connected graph $G = (V, E)$ with a depth-first search tree $T \subseteq E$. Let us be given a cost function $c : (E - T) \to \mathbb{R}_+$ on the edges outside T. Give a polynomial time algorithm for finding a minimum-cost edge set $F \subseteq E - T$ such that $T + F$ is 2-node-connected. *(Hint: minimum cost circulations.)*

3. Let $G = (S \cup T, E)$ be a connected bipartite graph with $|S| = |T|$. Prove that the following two properties are equivalent: (i) Every edge in E is contained in a perfect matching. (ii) G can be constructed from a single edge by repeatedly adding odd length paths connecting two existing nodes in different color classes.

4. Prove that every minimal k-connected bipartite graph on n nodes contains at most $k(n - k)$ edges.

5. In a digraph, a feedback edge set is a set of edges intersecting every directed cycle. Prove that for planar graph, a minimum feedback edge set can be found in polynomial time.

6. Let $M = (S, \mathcal{F})$ be a matroid and B a basis. Let $y_1, \ldots, y_k \in S - B$, and let C_i denote the fundamental circuit of y_i with respect to B (the unique cycle in $B \cup \{y_i\}$). Let $x_i \in C_i \cap B$ for $i = 1, \ldots, k$ with the property that whenever $i < j$ then $x_i \notin C_j$. Prove that $B - \{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_k\}$ is also a basis.

7. A *cut* in a matroid is a minimal set of elements intersecting every basis. Prove that if A is a cut and C is a cycle, then $|A \cap C| \neq 1$.