Combinatorial optimization - Structures and Algorithms, *GeorgiaTech, Fall 2011* Lectures 7 – 9: Sep 13 – 20

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References are from the book *Connections in Combinatorial Optimization* by András Frank (**F**). All proofs covered in the lectures not included in the notes can be found in the book.

1 The Lucchesi-Younger theorem

Let $D = (V, E)$ be a directed graph, so that the underlying undirected graph is connected. A set of edges entering a set X is called a directed cut, if it has only incoming arcs, that is, $\delta(X) = 0$. X is called the **in-shore** of the cut. A **directed cut cover** or **dijoin** is a set F of arcs containing at least one arc from each directed cut.

It is easy to see that contracting the arcs in a dijoin results in a strongly connected digraph. The following property is less obvious.

Claim 1.1 (**F** Prop 9.7.1)**.** *Let $D = (V, E)$ be a digraph, and assume the underlying undirected graph is 2-EC. Let F be a dijoin of D minimal for containment. Then reorienting every arc in F yields a strongly connected orientation.*

Two directed cuts are disjoint, if they have no arcs in common. (Yet the in-shores may intersect. For example, given a star with nodes $t, v_1, \ldots, v_k$ and arcs $v_i t$, each arc $v_i t$ forms a directed cut with in-shore $V - \{v_i\}$. These cuts are pairwise disjoint.)

Theorem 1.2 (Lucchesi, Younger, **F** Thm 9.7.2)**.** *The minimum cardinality $\tau = \tau(D)$ of a dijoin equals the maximum number $\nu = \nu(D)$ of pairwise disjoint directed cuts.*

The proof in **F**, given by Lovász, uses uncrossing of directed cuts. The theorem also has a minimum cost version.

Theorem 1.3 (**F** Thm 9.7.4)**.** *Let us be given a cost function $c : E \to \mathbb{R}_+$ on the directed graph $G = (V, E)$. The minimum cost of a dijoin equals the maximum number of directed cuts so that each arc e is contained in at most $c(e)$ directed cuts.*

This can be obtained from the unweighted theorem by replacing every edge e by a path of length $c(e)$.

We shall exhibit a combinatorial algorithm for the (unweighted) Lucchesi-Younger theorem. For this, we need some prerequisites.

1.1 Detour: conservative weightings

Theorem 1.4. *Let $D = (V, E)$ be a directed graph with a cost function $c : E \to \mathbb{R}$. Then either there exists a negative cost cycle in E, or there exists a potential $\pi : V \to \mathbb{R}$ with $\pi(v) - \pi(u) \le c(uv)$ for every $uv \in E$.*

The proof is given by the following simple dynamic programming algorithm. For $k = 0, \ldots, n$, let $\pi_k(v)$ denote the minimum cost of a walk containing at most k edges, terminating in v. This can be computed by $\pi_0 \equiv 0$,

$$\pi_{k+1}(v) = \min\{0, \min_{uv \in E} c(uv) + \pi_k(u)\}.$$

If there exists a node v with $\pi_{n-1}(v) > \pi_n(v)$, the graph contains a negative cycle. Otherwise, $\pi(v) = \pi_n(v) = \pi_{n-1}(v)$ is a good potential.

If there is no negative cycle (and thus exists a good potential), then c is called a **conservative cost function**.

1.2 Algorithmic proof of the Lucchesi-Younger theorem (F Section 9.7.2)

In an undirected graph $G = (V, E)$ and a subset $X \subseteq V$, let $\sigma_0(X)$ denote the number of components of $G - X$.

Claim 1.5 (**F** Prop. 1.2.6)**.**

$$\sigma_0(X) + \sigma_0(Y) \le \sigma_0(X \cap Y) + \sigma_0(X \cup Y) + d_G(X, Y) \quad \forall X, Y \in V.$$

Proof. Induction on $|V| + |E|$. $X \cap Y$ can be deleted without modifying any term in the inequality, hence we may assume $X \cap Y = \emptyset$. Similarly, any arc induced in X, Y and in $V - (X \cup Y)$ can be contracted. Deleting an arc between X and Y does not change the LHS. On the RHS, $d_G(X, Y)$ decreases by 1, $\sigma_0(X \cup Y)$ does not change, and $\sigma_0(X \cap Y)$ increases by at most 1. Hence we may apply induction. Similar analysis works for the deletion of edges between $X \cup Y$ and $V - (X \cup Y)$. □

Let $D = (V, E)$ be the directed graph where we want to find a minimum dijoin. Assume we are given a dijoin F. Either we want to show that it is minimal or we want to find a smaller one.

For a directed cut X ($\delta(X) = 0$), let $\sigma_0(X)$ denote the undirected connected components of $D - X$ as above. Let $\sigma(X) = \sigma_0(X)$ if $X \ne \emptyset$ and $\sigma(\emptyset) = 0$.

For a dijoin F, $\rho(X) \ge \sigma(X)$ must hold. X is tight if this holds with equality.

Claim 1.6. *If X and Y are directed cuts, $X \cap Y \ne \emptyset$, then $X \cap Y$ and $X \cup Y$ are also directed cuts. If X and Y are tight, then also $X \cap Y$ and $X \cup Y$ are tight.*

Consequently, if $v \in V$ is contained in any tight directed cut, then there exists a unique minimal tight $T(v)$ containing v.

Let us define the directed graph $D' = (V, A)$ with a cost function c as follows. Let us add every edge $uv \in E$ to A with cost 1, and for every $uv \in F$, let us add vu with cost -1. Furthermore, add an edge uv with cost 0 whenever $u \in T(v)$.

If this weighting is conservative, then the potential π helps to identify $|F|$ disjoint directed cuts. If it is not, then changing F around an appropriately chosen negative cycle gives a smaller dijoin (see **F**).